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Six Bits for Nine Colored Quarks

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Abstract

The hypercomplex number system of the Dirac equation is used to generalize $SU(2)$ to the covering group of $SO(4)$. The basic representations in this number language suggest a parton model of 6 "bits" and 6 "antibits"; one with spin 0, two with spin $\frac{1}{2}$, and three with spin 1. The relationship of this to the special relativity group is also considered.

Introduction

Though the original simple quark theory had to be expanded (three colored triplets), and recent high-energy experimental results (Ellis, 1974) cast more doubt on the popular parton models, the parton concept may yet survive in some modified form. The $SU(2)$ description of isospin is certainly a useful concept. The empirical concept of strangeness introduced a generalization of the group. Thus, $SU(3)$ seems a rather "natural" extension, when $SU(2)$ is written in the standard 2 x 2 complex matrix form. There are well known difficulties with $SU(3)$ in relation to Lorentz covariance. This is very serious, since the nuclear force is strong and one should expect that a relativistic treatment is required to describe the partons that make up a proton, neutron, etc.

In this paper we would like to show that $SU(2)$ can be generalized in a 'natural' way, leading to the possibility of 6 basic partons and 6 antipartons, by casting it in hypercomplex number form. This generalization is compatible with special relativity if generalized to rotations in (4, 1) space-time. We have argued previously that this generalization is reasonable and connected with the existence of rest mass in nature. Here we shall only show how the group structure is formulated and how it suggests the number and spins of the partons. The quark model, though suggested by $SU(3)$, has much of its success without reference to the original motivating group. Our development may contain a similar pattern.

1. *SU(2) in Quaternion Form*

The Pauli matrices σ_k along with $\sigma_0 = 1$ form a basis for the complex quaternion number ring when complex coefficients are used. The hermitian conjugate A^* is supplemented by the quaternion conjugate $A^\ddagger$, where $\sigma_0{}^\ddagger \equiv \sigma_0$, $\sigma_k{}^\ddagger \equiv -\sigma_k$, and $C^\ddagger \equiv C$ for C a complex coefficient. We then have

$$A = A^\mu \sigma_\mu = A_R{}^\mu \sigma_\mu + iA_I{}^\mu \sigma_\mu \text{ and } (AB)^* = B^* A^*, (AB)^\ddagger = B^\ddagger A^\ddagger \quad (1.1)$$

The "spinor representation" of the rotation group now takes the form

$$\psi_a{}' \equiv R^\ddagger \psi_a, \psi_v{}' \equiv R^* \psi_v, RR^\ddagger \equiv 1\sigma_0, R^\ddagger \equiv R^*, \Rightarrow \psi_a = \psi_v \equiv \psi \quad (1.2)$$

An infinitesimal rotation satisfies

$$R = 1\sigma_0 + \delta^k \sigma_k, \delta\delta \approx 0, \Rightarrow \delta^\ddagger = -\delta = \delta^* \Rightarrow \text{generators } i\sigma_k \quad (1.3)$$

Since R has a 2×2 complex matrix representation, ψ has a 1×2 representation, and therefore two complex number parts (four real parameters). These parts are associated with ψ(proton) and ψ(neutron) in isospin.

For completeness, we remark that $\mathrm{LL}^\ddagger \equiv 1\sigma_0$ gives the Lorentz group, with generators $i\sigma_k$ and σ_k since $\mathrm{L}^* \neq \mathrm{L}^\ddagger$ in general. The Lorentz (space-time) transformation is given by $x' = x^{\mu\prime}\sigma_\mu = \mathrm{L}^* x \mathrm{L}$, and $(x|x) \equiv x^\ddagger x = x^\mu x_\mu \sigma_0$.

2. *Generalized Number System*

Because of rest mass, the Weyl equation, $i\hbar\partial^\mu \sigma_\mu \psi_a = 0$, is replaced by the Dirac equation, $i\hbar\partial^\mu (e_\mu)\psi_a = mc(if_0)\psi_a$, and the complex quaternion number system $\{\sigma_\mu, i\sigma_\mu\}$ is generalized to a direct product number system with basis elements $\{\sigma_0, i\sigma_1, \sigma_2, \sigma_3\} \otimes \{\sigma_0, i\sigma_1, i\sigma_2, i\sigma_3\}$; a 16-element number system with real coefficients. The basis can have the following matrix representation

$$(e_\mu) \equiv \begin{pmatrix} \sigma_\mu & 0 \\ 0 & \sigma_\mu{}^{\ddagger *} \end{pmatrix}, (ie_\mu) \equiv \begin{pmatrix} (i\sigma_\mu) & 0 \\ 0 & (i\sigma_\mu)^{\ddagger *} \end{pmatrix}, (f_\mu) \equiv \begin{pmatrix} 0 & \sigma_\mu{}^{\ddagger *} \\ \sigma_\mu & 0 \end{pmatrix},$$

$$(if_\mu) \equiv \begin{pmatrix} 0 & (i\sigma_\mu)^{\ddagger *} \\ (i\sigma_\mu) & 0 \end{pmatrix} \quad (2.1)$$

The hermitian conjugate, now written $(\ )^\dagger$, gives

$$(e_\mu)^\dagger = (e_\mu), (ie_\mu)^\dagger = -(ie_\mu), (f_0)^\dagger = (f_0), (f_k)^\dagger = -(f_k),$$
$$(if_0)^\dagger = (if_0), \text{ and } (if_k)^\dagger = -(if_k) \quad (2.2)$$

We generalize the quaternion conjugate $(\ )^\ddagger$ to $(\ )^\wedge$ as follows

$$(e_0)^\wedge \equiv (e_0), (e_k)^\wedge \equiv -(e_k), (ie_0)^\wedge \equiv (ie_0), (ie_k)^\wedge \equiv -(ie_k),$$
$$(f_\mu)^\wedge \equiv (f_\mu), \text{ and } (if_\mu)^\wedge \equiv -(if_\mu) \quad (2.3)$$

We then find that $(AB)^\dagger = B^\dagger A^\dagger$ and $(AB)^\wedge = B^\wedge A^\wedge$. In the usual Dirac matrix language, $A^\wedge$ is the same as $f_0 A^\dagger f_0$, f_μ is the same as γ_μ, and γ_5 is essentially (ie_0). We can show that the subalgebra $\{(e_\mu), (ie_\mu)\}$ is isomorphic to $\{\sigma_\mu, i\sigma_\mu\}$ so that $\{(f_\mu), (if_\mu)\}$ represents the extension of the complex quaternion number system. We therefore write $A = e + f$ to identify the e and f parts of the hypercomplex number A. By defining $e^\vee \equiv e^\wedge$ and $f^\vee \equiv -f^\wedge$ we can show that $(AB)^\vee = B^\vee A^\vee$ and that $A \equiv A^{\vee\wedge}$ means A has zero f part. These are all the properties of the number system that we shall need to generalize $SU(2)$. Other aspects have been elaborated elsewhere (Edmonds, 1974; 1975). We just mention, for completeness, that Lorentz transformations now take the form $x' = x^{\mu\prime}(e_\mu) = \mathrm{L}^\dagger x \mathrm{L}$, $\mathrm{LL}^\wedge \equiv 1(e_0)$, $\mathrm{L} = \mathrm{L}^{\wedge\vee}$. Dropping the restriction $\mathrm{L} = \mathrm{L}^{\vee\wedge}$ gives $x = x^\mu(e_\mu) + x^4(if_0)$ and (4, 1) space-time rotations—a 'natural' generalization.

3. *Generalized SU(2)*

With this beautiful machinery before us, it is now easy to see how $SU(2)$ should be generalized: $R \to S$, $(\ )^* \to (\ )^\dagger$, and $(\ )^\ddagger \to (\ )^\wedge$, where $S = S_a{}^\mu(e_\mu) + S_b{}^\mu(ie_\mu) + S_c{}^\mu(f_\mu) + S_d{}^\mu(if_\mu)$ with real coefficients (16 parameters). We now write the direct analogue of the $SU(2)$ development in equation (1.2) and (1.3). This gives

$$\psi_a{}' \equiv S^\wedge \psi,\ \psi_v{}' \equiv S^\dagger \psi_v,\ SS^\wedge \equiv 1(e_0),\ S^\wedge \equiv S^\dagger \Rightarrow \psi_a = \psi_v \equiv \psi \tag{3.1}$$

and

$$S \equiv 1(e_0) + \delta,\ \delta\delta \approx 0, \Rightarrow \delta^\wedge = -\delta = \delta^\dagger \Rightarrow \text{generators } (ie_k), (if_k) \tag{3.2}$$

Since $\sigma \leftrightarrow e$, we know that $\{(ie_k)\}$ generates rotations in 3-space. We note that

$$S \equiv \cos\theta(e_0) + \sin\theta(if_k) \Rightarrow SS^\wedge = (\cos^2\theta + \sin^2\theta)(e_0) = 1(e_0) \tag{3.3}$$

So S is a compact six parameter Lie group (covering $SO(4)$). We see now why Lorentz symmetry must be generalized in order to contain S as a subgroup. The $SU(2)$ subgroup of S is contained in $S = S^{\wedge\vee} \equiv R$.

4. *The Basic Representation*

Thus far we have only shown in an unusual language that 3-space rotations generalize easily to 4-space rotations. The representations, however, naturally suggest a basic set of partons as we now show.

From $\psi' = S^\dagger\psi$ and the fact that S has a 4×4 matrix representation, we see that ψ has four complex number parts (eight real parameters). This form is similar to the Dirac equation for spin $\frac{1}{2}$, so we "naturally" identify this representation with two particles and their antiparticles: p, n, and $\bar{p}, \bar{n}$.

Considering ψ as a hypercomplex number instead of a 1×2 matrix, we see that $(\psi')^{\dagger\wedge} = (S^\dagger\psi)^{\dagger\wedge} = S^\wedge\psi^{\dagger\wedge} = S^\dagger\psi^{\dagger\wedge}$. Therefore, $\psi = \pm\,\psi^{\dagger\wedge}$ gives the lowest dimensional representations in ψ:

$$(A)\ \psi = \psi^{\dagger\hat{}} \Rightarrow \psi = a^0(e_0) + a^k(ie_k) + b^0(f_0) + b^k(if_k)$$

$$\begin{aligned}(B)\ \psi = -\psi^{\dagger\hat{}} \Rightarrow \psi &= a^0(ie_0) + a^k(e_k) + b^0(if_0) + b^k(f_k)\\ &= (ie_0)[a^0(e_0) - a^k(ie_k) - b^0(f_0) + b^k(if_k)]\end{aligned} \tag{4.1}$$

We see that $(ie_0)A = B$ and $(ie_0)B = (ie_0)(ie_0)A = -A$. As mentioned earlier (ie_0) is essentially γ_5 in Dirac notation. Though the representations are not eigenstates of (ie_0), (which may mean $m \neq 0$), they are eigenstates of $-i(ie_3)$ and (f_0). We can display this by a change of basis. For representation A this gives

$$\begin{aligned}\psi = a_\pm^+([(e_0) \mp i(ie_3)] &+ (f_0)[(e_0) \mp i(ie_3)])\\ + b_\pm^+([-i(ie_1) \pm (ie_2)] &+ (f_0)[-i(ie_1) \pm (ie_2)])\\ + c_\pm^-([(e_0) \mp i(ie_3)] &- (f_0)[(e_0) \mp i(ie_3)])\\ + d_\pm^-([-i(ie_1) \pm (ie_2)] &- (f_0)[-i(ie_1) \pm (ie_2)])\end{aligned} \tag{4.2}$$

In the Dirac equation we take a similar structure for ψ, but, there we choose a basis which gives eigenstates of $-i(ie_3)$ and (if_0) for particles at rest. The (if_0) eigenstates (± 1) correspond to $E = \pm mc^2$, i.e., particle/antiparticle. We shall assume that the representations A and B above are not distinct enough to warrant extracting more than two particles and antiparticles from them, instead of four of each. If four were chosen, we would get an eight parton theory (and eight seems to be a magic number in parton and quaternion numerology).

The other basic representations, in this language, are

$$\phi' \equiv S^{\hat{}}\phi S,\ \phi^{\hat{}} \equiv \pm\phi \text{ and } A' \equiv S^\dagger A S,\ A^\dagger \equiv \pm A \tag{4.3}$$

But since $S^\dagger = S^{\hat{}}$, we see that there is no real difference between ϕ and A for the subgroup S, though there is for the group L. The lowest dimensional representations correspond to $A = \pm A^\dagger$ and $A = \pm A^{\hat{}}$, taken in all combinations. We find

$$\begin{aligned}&(\mathrm{I})\ A = A^\dagger = A^{\hat{}} \Rightarrow A = a(e_0) + b(f_0) = a(e_0) + (f_0)b(e_0)\\ &(\mathrm{II})\ A = -A^\dagger = -A^{\hat{}} \Rightarrow A = a^k(ie_k) + b^k(if_k) = a^k(ie_k) + (f_0)b^k(ie_k)\\ &(\mathrm{III})\ A = -A^\dagger = A^{\hat{}} \Rightarrow A = a^k(f_k) + b(ie_0) = (f_0)a^k(e_k) + (f_0)b(if_0)\\ &(\mathrm{IV})\ A = A^\dagger = -A^{\hat{}} \Rightarrow A = a^k(e_k) + b(if_0)\end{aligned} \tag{4.4}$$

We can show that $(E)(f_0) = (f_0)(E)^{\dagger\hat{}}$ for any hypercomplex (E). But $S = S^{\dagger\hat{}}$, therefore, S commutes with (f_0). Notice also that $(f_0)[(e_0) \pm (f_0)] = \pm 1\,[(e_0) \pm (f_0)]$ and $(f_0)[(ie_k) \pm (if_k)] = \pm 1\,[(ie_k) \pm (if_k)]$. Whereas, the representation III and IV give (f_0) [III] = [IV] and (f_0)[IV] = [III]. Since $x = x^\mu(e_\mu) + x^4(if_0)$, we see that these two representations correspond to the usual space-time representation of $SO(4)$, with fixed x^0.

Though these properties do not rigorously rule out the representations III and IV as regards partons, it may be possible to identify (f_0) with something like strangeness and/or desirable to have partons which are eigenstates of (f_0). At least we know (f_0) commutes with the group S, though its physical meaning is unclear here.

The group S has (ie_3) and (if_3) as its mutually commuting generator set (trunk). From the spin analogy, $-i(ie_3)$ would be the isospin operator. Then (if_3) should be an important physical operator also. Note that (f_0) commutes with both (ie_3) and (if_3), which is further evidence that (f_0) is an important physical operator. Also $(if_3) = (f_0)(ie_3)$, so (f_0) may be the generator replacing (if_3), physically.

Identifying mass with the fourth space dimension, indicates (if_0) is the particle/antiparticle operator. It commutes with (ie_3) but not with (if_3) or (f_0). For $x \equiv x^\dagger$, we find $x = x^\mu(e_\mu) + x^4(if_0) + x^5(f_0)$ and x^5 is invariant under L. It is, therefore, possible to instead identify mass with the invariant sixth dimension and (f_0) as the particle/antiparticle operator. This is an important question, yet to be resolved.

All these considerations lead us to *postulate* that only the I and II representations, along with one of the $\psi' = S^\dagger \psi$ representations, give the basic partons/antipartons. Since $L^{\wedge}(e_0)L = (e_0)$ and $L^\dagger(f_0)L = (f_0)$, we expect representation I to give spin 0 partons, $\lambda, \bar{\lambda}$. Since II and III can be combined to give the 10 component representation $A = -A^\dagger$ (spin 2) and III is spin 1, we expect that II is also spin 1 as a parton representation, giving α, β, γ, $\bar{\alpha}$, $\bar{\beta}$, and $\bar{\gamma}$. So, finally, we conclude that S produces one spin-0, two spin-1/2, and three spin 1 partons and their antipartons.

5. *Conclusion*

We have shown that $SU(2)$, with two partons p, n, can be generalized in a natural way, using the hypercomplex number system related to the Dirac–Clifford algebra. We obtain a set of six basic partons, which we label "bits," with one spin-0 "nickel," two spin-1/2 "dimes," and three spin-1 "quarters" (red, yellow, green). Any relativistically adequate parton model must describe all observed particles: leptons, and photons, as well as hadrons. This bit model has a variety of bits and antibits, so there is some hope that it can span such a wide range of particle types. It is also compatible with relativity if relativity proves to need generalization—the need of which I am convinced.

The next step is to analyze the direct product representations and particle families, in analogy with the procedures used on $SU(3)$ and similar parton models.

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